

Calculus Practice SAC

Worked Solutions

VCE Methods Units 1&2

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Section A – Technology-free

Question 1a

3 marks

Differentiate the position function $x(t) = pt^2 + qt^3$ with respect to t to find the velocity function $v(t)$.

$$v(t) = 2pt + 3qt^2$$

The train comes to rest at station **B** when $t = 30$, meaning $v(30) = 0$. Substitute $t = 30$ into the velocity expression.

$$2p(30) + 3q(30)^2 = 0$$

Simplify the equation to form a relation between p and q .

$$60p + 2700q = 0$$

$$p = -45q$$

[1M]

The train travels 450 metres in 30 seconds, so substitute $t = 30$ and $x(30) = 450$ into the position model.

$$p(30)^2 + q(30)^3 = 450$$

Expand the powers of 30 to produce a second equation in terms of p and q .

$$900p + 27\,000q = 450$$

[1M]

Substitute $p = -45q$ into the position equation to eliminate p .

$$900(-45q) + 27\,000q = 450$$

$$-13\,500q = 450$$

$$q = -\frac{1}{30}$$

$$q = -\frac{1}{30}$$

$$p = \frac{45}{30}$$

$$= \frac{3}{2}$$

[1A]

Question 1b

2 marks

Substitute $p = \frac{3}{2}$ and $q = -\frac{1}{30}$ into the velocity function to obtain the complete speed model.

$$v(t) = 3t - \frac{t^2}{10}$$

Differentiate $v(t)$ with respect to t to find the acceleration function $a(t)$.

$$a(t) = 3 - \frac{t}{5}$$

Maximum speed occurs where acceleration is zero, as this is the turning point of the velocity function.

$$3 - \frac{t}{5} = 0$$

Solve the linear equation for t to find the time at which maximum speed occurs.

$$t = 15 \quad [1M]$$

Substitute $t = 15$ into the velocity function and convert the unit from metres per second to kilometres per hour by multiplying by 3.6.

$$\begin{aligned} v(15) &= 45 - 22.5 \\ &= 22.5 \text{ m/s} \\ 22.5 \times 3.6 &= 81 \text{ km/h} \end{aligned} \quad [1A]$$

Question 1c

1 mark

Average speed is defined as total distance divided by total time. The train travels 450 metres in 30 seconds.

$$\frac{450}{30} = 15 \text{ m/s} \quad [1A]$$

Question 1d

1 mark

Substitute $t = 20$ into the acceleration function $a(t) = 3 - \frac{t}{5}$ derived in part b.

$$\begin{aligned} a(20) &= 3 - \frac{20}{5} \\ &= -1 \text{ m/s}^2 \end{aligned} \quad [1A]$$

Question 1e

2 marks

Set the velocity function equal to 20 to solve for the required times.

$$3t - \frac{t^2}{10} = 20$$

Multiply the entire equation by 10 and rearrange into standard quadratic form.

$$\begin{aligned} 30t - t^2 &= 200 \\ t^2 - 30t + 200 &= 0 \end{aligned} \quad [1M]$$

Factorise the quadratic expression into two linear binomials.

$$(t - 10)(t - 20) = 0$$

Solve for t from each factor.

$$t = 10 \text{ s or } 20 \text{ s} \quad [1A]$$

Section B – Technology-active

Question 2a

1 mark

Substitute $p = \frac{3}{2}$ and $q = -\frac{1}{30}$ into $x(t) = pt^2 + qt^3$ and set it equal to 126 m.

$$\frac{3t^2}{2} - \frac{t^3}{30} = 126$$

$$45t^2 - t^3 = 3780$$

$$t^3 - 45t^2 + 3780 = 0$$

[1A]

Question 2b

2 marks

State Newton's method formula $t_{n+1} = t_n - \frac{g(t_n)}{g'(t_n)}$, where $g(t) = t^3 - 45t^2 + 3780$.

$$g(t) = t^3 - 45t^2 + 3780$$

$$g'(t) = 3t^2 - 90t$$

Evaluate the function and its derivative at $t_0 = 12$.

$$\begin{aligned} g(12) &= 12^3 - 45(12)^2 + 3780 \\ &= -972 \end{aligned}$$

$$\begin{aligned} g'(12) &= 3(12)^2 - 90(12) \\ &= -648 \end{aligned}$$

[1M]

Substitute these values into Newton's formula to find t_1 .

$$\begin{aligned} t_1 &= 12 - \frac{-972}{-648} \\ &= 10.5 \end{aligned}$$

[1A]

Question 2c

1 mark

Solve with technology to find the root, $t \approx 10.4615$, then round to two decimal places.

$$t \approx 10.46 \text{ s}$$

[1A]

Question 2d

3 marks

The back of the train reaches the crossing when the train has travelled $126 + 25 = 151$ m. Set the displacement equation equal to 151.

$$\frac{3t^2}{2} - \frac{t^3}{30} = 151$$

Solve with technology to find the arrival time for the back of the train.

$$t \approx 11.6557$$

[1M]

Use the unrounded front arrival time $t \approx 10.4615$ from part c. The gates close 7 seconds before the front arrives and open 3 seconds after the back passes.

$$10.4615 - 7 = 3.4615$$

$$11.6557 + 3 = 14.6557$$

[1M]

Subtract the start time from the end time to find the total duration, then round to the nearest second.

$$14.6557 - 3.4615 = 11.1942 \\ \approx 11 \text{ s}$$

[1A]

Question 2e

1 mark

Substitute the time when the back of the train reaches the crossing into the velocity function from

part 1b, $v(t) = 3t - \frac{t^2}{10}$.

$$v(11.6557) = 3(11.6557) - \frac{(11.6557)^2}{10}$$

Evaluate the speed in metres per second.

$$v(11.6557) \approx 21.3816 \text{ m/s}$$

Multiply by 3.6 to convert from m/s to km/h, and round to one decimal place.

$$21.3816 \times 3.6 \approx 77.0 \text{ km/h}$$

[1A]

Question 3a

2 marks

Use the distance from A to B , which is 450 metres when $t = T$.

$$pT^2 + qT^3 = 450$$

The train comes to rest at B , so the velocity at $t = T$ is zero.

$$2pT + 3qT^2 = 0$$

[1M]

Solve this simultaneous system for p and q in terms of T with technology.

$$p = \frac{1350}{T^2}$$

$$q = -\frac{900}{T^3}$$

[1A]

Question 3b

2 marks

Differentiate the distance function with respect to t to find the velocity function $v(t)$.

$$v(t) = 2pt + 3qt^2$$

Differentiate velocity to obtain acceleration.

$$a(t) = 2p + 6qt$$

[1M]

Substitute the expressions for p and q in terms of T into $a(t)$.

$$a(t) = 2 \left(\frac{1350}{T^2} \right) + 6 \left(-\frac{900}{T^3} \right) t \\ = \frac{2700}{T^2} - \frac{5400t}{T^3}$$

The acceleration is linear in t , so its greatest magnitude is at an endpoint of the interval $[0, T]$.

$$\begin{aligned} a(0) &= \frac{2700}{T^2} \\ a(T) &= -\frac{2700}{T^2} \end{aligned} \quad [1A]$$

Question 3c

2 marks

Set the magnitude of the maximum acceleration equal to or less than 3.6 m/s^2 .

$$\frac{2700}{T^2} \leq 3.6$$

Rearrange the inequality to isolate T^2 .

$$T^2 \geq \frac{2700}{3.6}$$

Simplify the fraction on the right side.

$$T^2 \geq 750 \quad [1M]$$

Take the square root of both sides, keeping only the positive value since $T > 0$.

$$\begin{aligned} T &\geq \sqrt{750} \\ &= 5\sqrt{30} \\ &\approx 27.39 \end{aligned}$$

$T = 27$ gives a greatest acceleration of about 3.704 m/s^2 , which exceeds the limit, so test the next whole number upwards.

$$\frac{2700}{27^2} \approx 3.704 > 3.6, \frac{2700}{28^2} \approx 3.444 \leq 3.6, \text{ so } T = 28 \text{ s} \quad [1A]$$

Question 3d

2 marks

The acceleration is zero at the midpoint of the trip, $t = \frac{T}{2}$, where the velocity reaches its maximum.

$$v\left(\frac{T}{2}\right) = 2p\left(\frac{T}{2}\right) + 3q\left(\frac{T}{2}\right)^2$$

Substitute p and q in terms of T , then evaluate the maximum velocity expression.

$$\begin{aligned} v\left(\frac{T}{2}\right) &= pT + \frac{3qT^2}{4} \\ &= \left(\frac{1350}{T^2}\right)T + \frac{3}{4}\left(-\frac{900}{T^3}\right)T^2 \\ &= \frac{1350}{T} - \frac{675}{T} \\ &= \frac{675}{T} \end{aligned}$$

Substitute $T = 28$ into the maximum velocity formula.

$$v(14) = \frac{675}{28} \approx 24.107 \text{ m/s} \quad [1M]$$

Convert the speed from metres per second to kilometres per hour by multiplying by 3.6.

$$24.107 \times 3.6 \approx 87 \text{ km/h} \quad [1A]$$

Question 4a

2 marks

Velocity is the antiderivative of acceleration with respect to time t . Integrate $a(t) = -(0.3 + 0.02t)$ to find $v(t)$.

$$v(t) = \int -(0.3 + 0.02t) dt \quad [1M]$$

Evaluate the antiderivative and include the constant of integration c .

$$v(t) = -0.3t - 0.01t^2 + c$$

Use the initial condition that the train approaches station C at 25 m/s when $t = 0$, so $v(0) = 25$.

$$25 = -0.3(0) - 0.01(0)^2 + c$$

Substitute $c = 25$ back into the velocity function.

$$v(t) = 25 - 0.3t - 0.01t^2 \text{ m/s} \quad [1A]$$

Question 4b

2 marks

The train stops when its velocity is zero, so set the velocity expression equal to zero.

$$25 - 0.3t - 0.01t^2 = 0 \quad [1M]$$

Multiply the equation by -100 to clear the decimals.

$$t^2 + 30t - 2500 = 0$$

Apply the quadratic formula to solve for t , noting that time must be positive.

$$\begin{aligned} t &= \frac{-30 \pm \sqrt{30^2 - 4(1)(-2500)}}{2(1)} \\ &= \frac{-30 \pm \sqrt{900 + 10000}}{2} \\ &= \frac{-30 \pm \sqrt{10900}}{2} \\ &= -15 \pm 5\sqrt{109} \end{aligned}$$

Reject the negative time value because time starts at $t = 0$ when the brakes are applied.

$$\begin{aligned} t &= -15 + 5\sqrt{109} \\ &\approx 37.20 \text{ s} \end{aligned} \quad [1A]$$

Question 4c

2 marks

Displacement is the antiderivative of velocity, so integrate $v(t) = 25 - 0.3t - 0.01t^2$ with respect to t .

$$s(t) = \int (25 - 0.3t - 0.01t^2) dt \quad [1M]$$

Evaluate the integral, noting that displacement starts at zero when the brakes are applied so the constant of integration is zero.

$$s(t) = 25t - 0.15t^2 - \frac{t^3}{300}$$

Substitute the stopping time from part b, $t = -15 + 5\sqrt{109} \approx 37.2015$, into the distance function to find the total stopping distance.

$$s(-15 + 5\sqrt{109}) = 25(37.2015) - 0.15(37.2015)^2 - \frac{(37.2015)^3}{300}$$

Evaluate using technology and round to the nearest metre.

$$\begin{aligned} s(37.2015) &\approx 550.8278 \\ &\approx 551 \text{ m} \end{aligned} \quad [1A]$$

Question 4d

2 marks

Set the distance function equal to half of the unrounded stopping distance from part c, restricting t to the time interval until the train stops.

$$25t - 0.15t^2 - \frac{t^3}{300} = \frac{550.8278}{2}, \quad 0 \leq t \leq 37.2015 \quad [1M]$$

Solve the equation for t using technology.

$$t \approx 12.14$$

Round the result to one decimal place as requested.

$$t \approx 12.1 \text{ s} \quad [1A]$$

Question 5a

2 marks

Equate the distance expressions for the train and the car.

$$\frac{3t^2}{2} - \frac{t^3}{30} = 16t$$

Divide both sides by t since $t > 0$, multiply by 30, and rearrange into standard quadratic form.

$$\begin{aligned} 45t - t^2 &= 480 \\ t^2 - 45t + 480 &= 0 \end{aligned} \quad [1M]$$

Apply the quadratic formula to solve for t .

$$\begin{aligned} t &= \frac{-(-45) \pm \sqrt{(-45)^2 - 4(1)(480)}}{2(1)} \\ &= \frac{45 \pm \sqrt{105}}{2} \end{aligned}$$

Both solutions lie in the interval $(0, 30]$ because $\sqrt{105} \approx 10.25$.

$$t = \frac{45 - \sqrt{105}}{2} \text{ s or } \frac{45 + \sqrt{105}}{2} \text{ s} \quad [1A]$$

Question 5b

2 marks

The separation between the train and the car is $D(t) = x(t) - 16t$. A stationary point occurs where the train's velocity equals the car's speed.

$$x'(t) = 16$$

Differentiate $x(t) = \frac{3t^2}{2} - \frac{t^3}{30}$ with respect to t .

$$3t - \frac{t^2}{10} = 16$$

Multiply through by -10 and rearrange into standard quadratic form.

$$t^2 - 30t + 160 = 0$$

$$(t - 15)^2 - 65 = 0$$

The train is only ahead of the car between the two intersection times from part a, so reject

$$15 - \sqrt{65} \approx 6.94.$$

$$t = 15 + \sqrt{65} \approx 23.06 \quad [1M]$$

Substitute the exact time $t = 15 + \sqrt{65}$ into $x(t) - 16t$ to determine the maximum distance ahead.

$$x(15 + \sqrt{65}) - 16(15 + \sqrt{65}) \approx 19.94 \approx 20 \text{ m} \quad [1A]$$

Question 5c

3 marks

The train and car are level when their positions are equal.

$$\frac{3t^2}{2} - \frac{t^3}{30} = ut$$

Divide both sides by t , since $t > 0$, to express u in terms of t .

$$u = \frac{3t}{2} - \frac{t^2}{30}, \quad 0 < t \leq 30 \quad [1M]$$

Differentiate this expression with respect to t to find the location of the turning point.

$$\frac{d}{dt} \left(\frac{3t}{2} - \frac{t^2}{30} \right) = \frac{3}{2} - \frac{t}{15}$$

$$\frac{3}{2} - \frac{t}{15} = 0$$

$$t = 22.5$$

Evaluate the function at the turning point $t = 22.5$ and at the right endpoint $t = 30$.

$$\frac{3(22.5)}{2} - \frac{22.5^2}{30} = \frac{135}{8}$$

[1M]

$$\frac{3(30)}{2} - \frac{30^2}{30} = 15$$

As $t \rightarrow 0^+$, the expression approaches 0. It rises to a maximum of $\frac{135}{8}$ at $t = 22.5$ and falls to 15 at $t = 30$. A horizontal line $y = u$ intersects this curve once either at the local maximum or strictly between 0 and 15.

$$0 < u < 15 \text{ or } u = \frac{135}{8}$$

[1A]